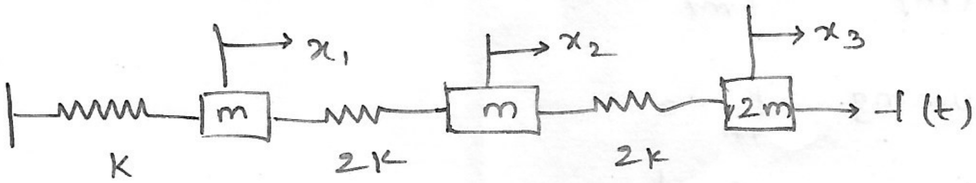
-! Advanced Structural Dynamics !-

* Now we have System such that



$K = 1.4 \times 10^5$ N/m $\quad m = 20$ kg.

and $f(t) = \begin{cases} 2000t & 0 \le t \le 0.1 \\ 250t + 175 & 0.1 < t \le 0.5 \\ 300 & t \ge 0.5 \end{cases}$

Now the equation of motion for this will be

$$m\ddot{x}_1 + Kx_1 + 2K(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + 2K(x_2 - x_1) + 2K(x_2 - x_3) = 0$$

$$2m\ddot{x}_3 + 2K(x_3 - x_2) = F(t)$$

writing in the form of Matrix Notation

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \qquad F = \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$$

$$[K] = \begin{bmatrix} 3K & -2K & 0 \\ -2K & 4K & -2K \\ 0 & -2K & 2K \end{bmatrix}$$

Now finding the ~~Char.~~ Natural frequency of the system

$$[K] - \omega^2 [m] = 0$$

The resulting charecteristic equation is

$$4\left(\frac{k}{m}\right)^3 - 26\left(\frac{k}{m}\right)^2 \omega^2 - 16\left(\frac{k}{m}\right)\omega^4 + 2\omega^6 = 0$$

Now assuming $\frac{k}{m} = a$

and

$\omega^2 = b$

and $x = (b/a)$

we get The result

$$-2x^3 + 16x^2 - 26x + 4 = 0$$

and $x = 2$

$3 + 2\sqrt{2}$

$3 - 2\sqrt{2}$

Now calculating Natural frequencies with The substitution

we get

$\omega_1 = 34.6556$

$\omega_2 = 201.9876$

$\omega_3 = 118.3216$

Now Calculating Modal Matrix

$$[[K] - \omega^2[M]]\,[u] = 0$$

$$u_1 = \begin{bmatrix} 1.00 \\ -1.4142 \\ -1.7071 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 1.4142 \\ -0.2929 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1.000 \\ -0.500 \\ 0.5000 \end{bmatrix}$$

Normalising the Mass Matrix.

$$\begin{bmatrix} 0.0057 & 0.0158 & 0.0286 \\ -0.0080 & 0.0223 & -0.0143 \\ -0.0097 & -0.0046 & 0.0143 \end{bmatrix}$$

Transpose of Matrix

$$\phi^T = \begin{bmatrix} 0.0057 & -0.0080 & -0.0097 \\ 0.0158 & 0.0223 & -0.0046 \\ 0.0286 & -0.0143 & 0.0143 \end{bmatrix}$$

* Now

$$Q = [\phi^T][F] = \begin{bmatrix} -0.0097 \\ -0.0046 \\ 0.0143 \end{bmatrix} f(t)$$

Now we have to solve the equation

$$\ddot{\eta}_p + \omega_p^2 \eta_p = Q_p$$

and then the solution for the equation is given

$$\ddot{\eta}_p + \omega_p^2 \eta_p = Q_p$$

Taking Laplace Transform

$$L[\ddot{\eta}_p] + \omega_p L[\eta_p] = L[Q_p]$$

$$\Rightarrow \quad s^2 F(s) - s\,\eta_p(0) - \dot{\eta}_p(0) = L[Q_p]$$

Now $\eta_p(0) = 0$
$\dot{\eta}_p(0) = 0$ for the System

$$s^2 F(s) = L[X_p \cancel{F}(t)]$$

$\Rightarrow X_p \times \mathcal{L}[f(t)]$

$$\int_0^{0.1} e^{-st}\, 2000t \, dt + \int_{0.1}^{0.5} e^{-st} (175+250t)\,dt + \int_{0.5}^{\infty} e^{-st}\, 300 \, dt$$

$$\Rightarrow 2000 \int_0^{0.1} t\, e^{-st} \, dt \quad — ①$$

$$\left[t \frac{e^{-st}}{-s}\right]_0^{0.1} - \int_0^{0.1} \frac{e^{-st}}{-s} \, dt$$

$$\frac{0.1 \times e^{-s0.1}}{-s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{0.1}$$

$$\frac{0.1\, e^{-0.1s}}{-s} - \frac{1}{s^2}\left[e^{-s0.1} - 1\right]$$

$$\frac{1}{s^2} - \frac{e^{-0.1s}}{s^2} - \frac{0.1\, e^{-0.1s}}{s}$$

$$= \frac{1}{s^2} - e^{-s/10}\left[\frac{1}{s^2} - \frac{1}{10s}\right] \Rightarrow \frac{1}{s^2} - e^{-s/10}\left[\frac{10s-}{10s}\right.$$

Now for second part

$$250 \int_{0.1}^{0.5} t\, e^{-st} + 175 \int_{0.1}^{0.5} e^{-st}$$

$$\quad ① \qquad\qquad ②$$

Thus

$$250\left[\left[\frac{t\, e^{-st}}{-s}\right]_{0.1}^{0.5} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{0.1}^{0.5}\right] + 175\left[\frac{e^{-st}}{-s}\right]_{0.1}^{0.5}$$

$$\Rightarrow 250\left[\frac{0.5\, e^{-s/2}}{-s} - \frac{0.1\, e^{-s/10}}{-s}\right] - \frac{250}{s^2}\left[e^{-t/2} - e^{-t/10}\right]$$

Now

$$\cancel{\frac{-125}{s} e^{-s/2}} + \frac{25}{s} e^{-s/10} - \cancel{\frac{250}{s^2} e^{-s/2}} + \frac{250}{s^2} e^{-s/10}$$

$$+ \cancel{\frac{175\, e^{-s/2}}{-s}} + \frac{175\, e^{-s/10}}{s}$$

Combining all together we get

$$\left[-\frac{125}{s} - \frac{175}{s} - \frac{250}{s^2}\right] e^{-s/2} + \left[\frac{25}{s} + \frac{175}{s} + \frac{250}{s^2}\right] e^{-s/10}$$

$$= \frac{200s + 250}{s^2} e^{-s/10} - \frac{300s + 250}{s^2} e^{-s/2}$$

for third part

$$300 \int_{0.5}^{\infty} e^{-st}\, dt \rightarrow 300 \left[\frac{e^{-st}}{-s}\right]_{0.5}^{\infty}$$

$$= 300 \cdot \frac{e^{-s/2}}{s}$$

Combining every thing. we will get that

$$\frac{1}{s^2} + e^{-s/10}\left[\frac{200s + 250 + 1 - s/10}{s^2}\right] + e^{-s/2}\left[\frac{300s - 300s + 250}{s^2}\right]$$

$$x_p \Rightarrow \left[\frac{1}{s^2} + e^{-s/10}\left[\frac{199.9s + 251}{s^2}\right] + e^{-s/2}\left[\frac{250}{s^2}\right]\right] x_p$$

equation is

$$\left[\mathcal{L}[x_r] = \frac{x_p}{s^4}\left[1 + e^{-s/10}(199.9s + 251) + e^{-s/2}\, 250\right]\right]$$

For the Solution we have to find inverse Laplace Transfo

$$\frac{\eta_p}{x_p} = \mathcal{L}^{-1}\left[\frac{1}{s^4}\right] + \mathcal{L}^{-1}\left[e^{-s/10}\left[\frac{199.9s+251}{s^4}\right]\right] \quad — ①$$

$$+ \mathcal{L}^{-1}\left[e^{-s/2}\ \frac{250}{s^4}\right] \quad — ②$$

we have That

$$\mathcal{L}\left[f(t-\alpha)\ 1(t-\alpha)\right] = e^{-\alpha s}\ F(s)$$

$$f(t-a)\ 1(t-\alpha) = \mathcal{L}^{-1}\left[e^{-\alpha s}\ F(s)\right]$$

where $\mathcal{L}^{-1}\left[F(s)\right] = f(t)$

for ①

$$f_1\left(t-\tfrac{1}{10}\right)\ 1\left(t-0^{1}/_{10}\right)$$

$1(t)$ – unit function

for ②

$$f_2\left(t-\tfrac{1}{2}\right)\ 1\left(t-\tfrac{1}{2}\right)$$

Now Solving $f(t)$ for ①

$$f_1(t) = \mathcal{L}^{-1}\left[\frac{199.9s+251}{s^4}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{199.9}{s^3} + \frac{251}{s^4}\right]$$

Now with this

$$\frac{0}{s} + \frac{0}{s^2} + \frac{199.9}{s^3} + \frac{251.0}{s^4}$$

we have That

$$\mathcal{L}^{-1} \frac{199.9}{s^3} = \frac{199.9}{2!} \mathcal{L}^{-1}\left[\frac{2!}{s^{2+1}}\right]$$

$$= \frac{199.9}{2!} t^2$$

Similarly

$$\frac{251.0}{3!} t^3$$

and similarly other terms.

$$\frac{250}{3!} t^3 \quad - (2)$$

First term.

$$\frac{1}{3!} t^3 \quad - (I)$$

So
$$\frac{\eta_p}{X_p} = \frac{t^3}{3!} + \frac{250}{3!}\left(t - \frac{1}{2}\right)^3 u\left(t - \tfrac{1}{2}\right) + \left[\frac{199.9}{2!}\left(t - \frac{1}{10}\right)^2 + \frac{251}{3!}\left(t\right.\right.$$
$$\times\, u\left(t - \tfrac{1}{10}\right)$$

finally

$$\left[\ \eta_p = X_p\left[\frac{t^3}{6} + \frac{250}{6}\left(t - \tfrac{1}{2}\right)^3 u\left(t - \tfrac{1}{2}\right) + \left[\frac{199.9}{2}\left(t - \tfrac{1}{10}\right)^2 + \frac{251}{6}\left(t - \frac{1}{10}\right)^3\right] u\left(t - \tfrac{1}{10}\right)\right]\right.$$

where X_p is Cofficient ~~Amplitude~~

~~u(t)~~ a $u(t)$ - Unit function

Heravyside function

The responce is

$$\begin{bmatrix} 0.2806 \\ -0.2296 \\ 0.3189 \end{bmatrix} \times 10^{-3} \left[\frac{t^3}{6} + u(t-1/2)\frac{250}{6}(t-1/2)^3 + \left[\frac{199}{2}(t-1/10)^2 + \frac{251}{6}\left(t-\frac{1}{10}\right)^3\right] u(t\right.$$

is the responce

thus

$$u(t) = 0.3699 \times 10^{-3} \left[\frac{t^3}{6} + u(t-1/2)\frac{250}{6}(t-1/2)^3 + \left[\frac{199.9}{2}(t-1/10)^2 + \frac{251}{6}(t-1/10)^3\right] u\left(t-\frac{1}{10}\right.\right.$$